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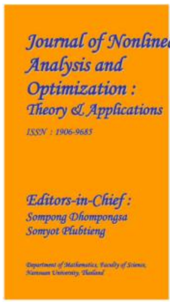

ARITHMETIC PROPERTIES OF NUMBER TRIANGLE

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Abstract
Several patterns of numbers can be revealed by arranging them in triangular or some particular fashion. In this paper, we will introduce a number triangle containing natural numbers arranged in a specific way and investigate some interesting arithmetic properties. Using this number triangle, we have proved five interesting results which help us to understand the hidden structure and patterns of numbers.

Keywords: Number Triangle, Row Sum Property, Centered Numbers, Alternating Sum Property.

1. Introduction

Several interesting number triangles have emerged and each such triangle has explored important properties involved with numbers in the triangle. In this paper, we will construct a simple number triangle using natural numbers arranged in each row in an ascending order. This number triangle has very interesting mathematical properties of natural numbers constituting the triangle.

The sequence 1,3,5,7,9,… representing odd natural numbers occurs in many places of mathematics. These numbers can be represented in the form of triangular grid of points where the first row contains a single number and each subsequent row contains two more numbers than the previous row as shown in Figure 1.

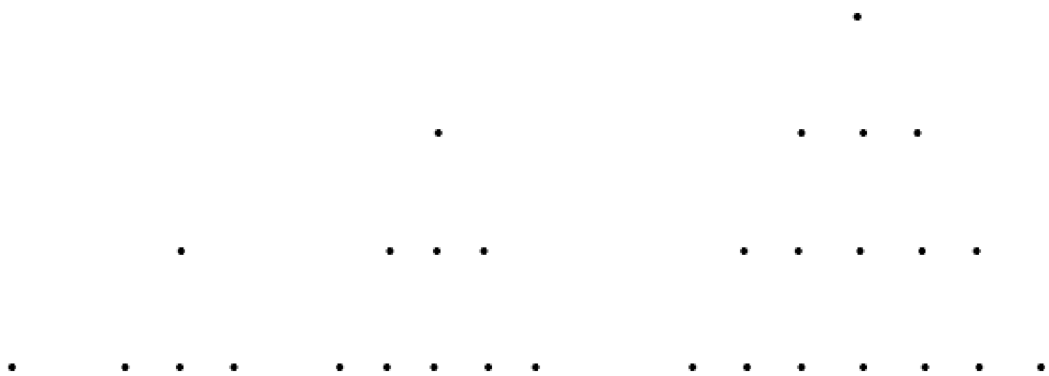

Figure 1

2. Construction of Number Triangle

First, let us consider the following number triangle constructed with successive natural numbers.

1

2 3 4

5 6 7 8 9

10 11 12 13 14 15 16

17 18 19 20 21 22 23 24 25

. . . .

Figure 2

If we consider m as row number, where m is a natural number then we observe from Figure 2, that row m contains $2m - 1$ successive natural numbers with $m^2 - 2m + 2$ as first number and m^2 as last number. Further we notice from Figure 2, that the number located in m^{th} row and n^{th} position (when read from left to right) where $m \geq 1$, $1 \leq n \leq 2m - 1$ is given by $A_{m,n} = (m-1)^2 + n$ (2.1)

Using equation (2.1), we now present some results regarding the number triangle presented in Figure 2.

Properties of Number Triangle

3. Theorem 1 (Row Sum Property)

For $m \geq 1$, the sum of row entries of number triangle of Figure 2 is given by

$$\sum_{n=1}^{2m-1} A_{m,n} = (2m-1)(m^2 - m + 1) \qquad (3.1)$$

Proof: Using (2.1) we have

$$\sum_{n=1}^{2m-1} A_{m,n} = \sum_{n=1}^{2m-1} (m-1)^2 + n$$

$$= (2m-1)(m-1)^2 + \sum_{n=1}^{2m-1} n$$

$$= (2m-1)(m-1)^2 + (2m-1)(m)$$

$$= (2m-1)\{(m-1)^2 + m\}$$

$$= (2m-1)(m^2 - m + 1)$$

This completes the proof.

4. Theorem 2 (Centered Numbers)

The centered numbers in m^{th} row of number triangle in Figure 2 is given by

$$A_{m,m} = 2T_{m-1} + 1 \qquad (4.1)$$

where T_{m-1} is the $(m-1)^{th}$ triangular number.

Proof: In Figure 2, the middle term of each row are called centered numbers.

Using (2.1) we have

$$A_{m,m} = (m-1)^2 + m$$

$$= m^2 - 2m + 1 + m$$

$$= m^2 - m + 1$$

$$= m(m-1) + 1$$

$$= 2T_{m-1} + 1$$

The centered numbers are one more than twice the triangular numbers.

This completes the proof.

5.Theorem 3 (Alternating Sum Property)

The alternating sum of row entries of m^{th} row of number triangle of Figure 2 is the centered number given by

$$\sum_{n=1}^{2m-1}(-1)^{n-1}A_{m,n} = 2T_{m-1}+1 \qquad (5.1)$$

Proof: Using (2.1) we have

$$\sum_{n=1}^{2m-1}(-1)^{n-1}A_{m,n} = \sum_{n=1}^{2m-1}(-1)^{n-1}\{(m-1)^2+n\}$$

Where $m \geq 1$, $1 \leq n \leq 2m-1$.

$$= \sum_{n=1}^{2m-1}(-1)^{n-1}(m-1)^2 + \sum_{n=1}^{2m-1}(-1)^{n-1}$$

$= (m-1)^2\{1-1+1 \cdots\cdots\cdots +(-1)^{2m-2}\} + \{1-2+3\cdots\cdots\cdots+(-1)^{2m-2}(2m-1)\}$ (5.2)

For $m \geq 1$, $(-1)^{2m-2} = 1$, then from (5.2) we get

$$\sum_{n=1}^{2m-1}(-1)^{n-1}A_{m,n} = (m-1)^2(1)+\{1-2+3\cdots+(-1)^{2m-2}(2m-1)\} \qquad (5.3)$$

Now, $2m-1$ is an odd number for $m \geq 1$, and so we get

$$\{1-2+3\cdots\cdots\cdots(-1)^{2m-2}(2m-1)\} = \frac{2m-1+1}{2} = \frac{2m}{2} = m$$

Hence, from (5.3) we get

$$\sum_{n=1}^{2m-1}(-1)^{n-1}A_{m,n} = (m-1)^2+m$$

$= (m-1)^2 + m$

$= m^2 - 2m + 1 + m$

$= m^2 - m + 1$

$= m(m-1) + 1$

$= 2T_{m-1} + 1$

This completes the proof.

6.Theorem 4 (Recurrence Property)

For $m \geq 1$ and $1 \leq n \leq 2m-1$, $A_{m+1,n+1} - A_{m,n} = 2m$. (6.1)

Proof: Using (2.1) we have

$$A_{m,n} = (m-1)^2 + n$$

Then

$A_{m+1,n+1} = \{((m+1)-1)^2 + (n+1)\}$

$= m^2 + n + 1$

$= m^2 + n + 1 - 2m + 2m$

$= (m-1)^2 + n + 2m$

$= A_{m,n} + 2m$

$A_{m+1,n+1} - A_{m,n} = 2m.$

This completes the proof.

7.Theorem 5

For $m \geq 1$ and $1 \leq n \leq 2m-1$, $A_{m,m-1} = 2T_{m-1}$ (7.1)

Proof: Using (2.1) we have

$A_{m,m-1} = (m-1)^2 + m - 1$

$= m^2 - 2m + 1 + m - 1$

$= m^2 - m$

$= m(m-1)$

$= 2T_{m-1}$

The numbers to the left of centered numbers are twice the triangular numbers.
This completes the proof.

8.Theorem 6

The left extreme diagonals of the number triangle in Figure 2 is given by

$$A_{m,1} = m^2 - 2m + 2 \qquad (8.1)$$

The right extreme diagonals of the number triangle in Figure 2 is given by

$$A_{m,2m-1} = m^2 \qquad (8.2)$$

Proof: Using (2.1) we have

$A_{m,1} = (m-1)^2 + 1$

$= m^2 - 2m + 1 + 1$

$= m^2 - 2m + 2$

The left extreme diagonals of the number triangle are one more than the perfect square.
Thus (8.1) is proved.
Using (2.1) we have

$A_{m,2m-1} = (m-1)^2 + 2m - 1 = m^2$

The right extreme diagonals of the number triangle are perfect square. Hence (8.2) is proved.
This completes the proof.

9.Theorem 7

The four numbers $A_{m,n}, A_{m,n+1}, A_{m+1,n+1}$ and $A_{m+1,n+2}$ of the number triangle located in the m^{th} and $(m+1)^{th}$ row from the vertices of the square satisfying the property given by

$$\{A_{m,n+1} \times A_{m+1,n+1}\} - \{A_{m,n} \times A_{m+1,n+2}\} = 2m \qquad (9.1)$$

Where $m \geq 2$ and $1 \leq n \leq 2m - 1$.

Proof: Using (2.1) we have

$\{A_{m,n+1} \times A_{m+1,n+1}\} - \{A_{m,n} \times A_{m+1,n+2}\}$

$= [(m-1)^2 + n + 1]\,[m^2 + n + 1] - [(m-1)^2 + n]\,[m^2 + n + 2]$

$= [m^2 - 2m + n + 2]\,[m^2 + n + 1] - [m^2 - 2m + n + 1]\,[m^2 + n + 2]$

$= -2m + 4m = 2m$

This completes the proof.

10.Theorem 8

The determinant of a **2**×**2** square matrix formed by entries located in the m^{th} and $(m+1)^{th}$ row is $-2m$.

Proof: Consider the four numbers $A_{m,n}, A_{m,n+1}, A_{m+1,n+1}$ and $A_{m+1,n+2}$ of the number triangle located in the m^{th} and $(m+1)^{th}$ row. The determinant of a 2×2 square matrix formed by these entries is given by

$$\{A_{m,n} \times A_{m+1,n+2}\} - \{A_{m,n+1} \times A_{m+1,n+1}\}$$

Using (2.1) we have

$\{A_{m,n} \times A_{m+1,n+2}\} - \{A_{m,n+1} \times A_{m+1,n+1}\}$

$= [(m-1)^2 + n]\,[m^2 + n + 2] - [(m-1)^2 + n + 1]\,[m^2 + n + 1]$

$= [m^2 - 2m + n + 2]\,[m^2 + n + 2] - [m^2 - 2m + n + 2]\,[m^2 + n + 1]$

$= -4m + 2m = -2m$

This completes the proof.

11. Conclusion

By introducing simple number triangle (as in Figure 2) consisting natural numbers arranged successively in each row such that m^{th} row contains $2m-1$ successive natural numbers, we had proved interesting results in this paper. These results will explore the arithmetic properties of number triangle. In this paper, we have proved eight theorems using the general term of the number triangle.

In theorem 1, we have proved the row sum property. In Theorem 2, centered numbers are derived in terms of triangular numbers. Theorem 3 establishes the relationship between the alternating sum of row entries and centered numbers. In theorem 4, recurrence property is proved. In theorem 5, we have proved that the numbers to the left of centered numbers are twice the triangular numbers. In theorem 6, we have proved results about diagonal numbers. In theorem 7, we have proved the result called square property since the four entries considered form the vertices of a square of the number triangle. In theorem 8, we have proved a result containing entries from the number triangle forming particular 2×2 determinants. Both figurate and arithmetic aspects of number triangle are attractive center of research for the mathematicians. The results derived in this paper will provide more insights to understand various properties of numbers.

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